ON THE STABILITY OF THE TRIANGULAR LIBRATION POINTS IN THE CIRCULAR BOUNDED THREE-BODY PROBLEM

PMM Vol. 33, No. 1, 1969, pp. 112-116

A.P. MARKEEV
(Moscow)

(Received July 26, 1968)

The results of a study of the stability of the equilibrium position of an autonomous Hamiltonian system with two degrees of freedom are presented. It is shown that the triangular libration points are stable in the first approximation for all ratios of the masses in the stability range, with the exception of certain specific ratios for which they are unstable.

1. In 1772 Lagrange [1] showed that the differential equations of motion of the three-body problem have a particular solution corresponding to the triangular libration points: the three bodies form an equilateral triangle which rotates in its plane about the center of mass of the bodies.

In the bounded circular problem two bodies (the body S of mass m_1 and the body J of mass m_2) move along circular orbits about their common center of mass O with the constant angular velocity n. The third body moves in the plane OSJ without affecting the motion of the bodies S, and J.

We know [2-4] that for $(m_1 + m_2)^2 > 27m_1m_2$ the triangular libration points in the bounded circular three-body problem are stable in the first approximation. Making use of the results of [5], Leontovich shows [6] that the libration points are stable for all m_1, m_2 in the range $(m_1 + m_2)^2 > 27 m_1m_2$ with the possible exception of a set of zero Lebesgue measure. In [7] the theorem of Arnol'd on the stability of the equilibrium position of an autonomous Hamiltonian system with two degrees of freedom [8] is used to show that the libration points are stable for all mass ratios m_1 / m_2 in the range $(m_1 + m_2)^2 > 27m_1m_2$ with the possible exception of three ratios for which the Arnol'd theorem does not hold.

We shall solve the problem of stability of the triangular libration points for all mass ratios satisfying the condition of stability in the first approximation.

2. Let us consider the stability of the equilibrium position of a canonical system with two degrees of freedom.

1°. Let the origin be the equilibrium position of the system

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2) \tag{2.1}$$

Here H is a Hamiltonian which is independent of t, analytic in q_i, p_i and can be expanded in a series

$$H = H_2 + H_3 + H_4 + \ldots + H_k + \ldots \tag{2.2}$$

where H_k is a homogeneous function of degree k in q_i, p_i.

If H_2 is a function of fixed sign, then the equilibrium position is stable by virtue of the Liapunov theorem [9]. On the other hand, if H_2 is not a function of fixed sign, then stability can be investigated by means of the Arnol'd theorem [8]. Let Hamiltonian (2.2) satisfy the three following conditions:

1) the characteristic equation of the linearized system has the purely imaginary roots $\pm i\omega_1,\ \pm i\omega_2$;

2) the frequencies $\omega_1,\ \omega_2$ satisfy the inequalities

$$k_1\omega_1 + k_2\omega_2 \neq 0 \qquad \text{for } 0 < |k_1| + |k_2| < 4 \tag{2.3}$$

where k_1 and k_2 are integers

3) the inequality

$$c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0 \tag{2.4}$$

is fulfilled.

Fulfillment of these conditions ensures the stability of the equilibrium position.

The formulation of the theorem assumes that Hamiltonian (2.2) has been reduced to the form

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + O((r_1+r_2)^{5/2}), (2r_i = p_i^2 + q_i^2) \tag{2.5}$$

Such a reduction is possible [10] if condition (2.3) is fulfilled.

For complete investigation of the problem of stability of the equilibrium position of system (2.1) we must consider the cases in which conditions (2.3) or (2.4) are not fulfilled. Let $\omega_1 > \omega_2$. Inequalities (2.3) are then violated for $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$. Stability in these resonance cases is investigated in [11]. We shall now cite the basic results necessary for our subsequent investigation.

2°. With suitable choice of the variables $q_i,\ p_i$ in the case $\omega_1 = 2\omega_2$ Hamiltonian (2.2) becomes

$$H = 2\omega_2 r_1 - \omega_2 r_2 - \sqrt{(x_{1002}^2 + y_{1002}^2)\,\omega_2}\, r_2 \sqrt{r_1} \sin(\varphi_1 + 2\varphi_2) + O((r_1 + r_2)^2) \tag{2.6}$$

Here $x_{1002},\ y_{1002}$ are constants which depend on the coefficients of the forms H_2 and H_3 in expansion (2.2), and

$$q_i = \sqrt{2r_i}\sin\varphi_i, \qquad p_i = \sqrt{2r_i}\cos\varphi_i\ (i = 1, 2) \tag{2.7}$$

If $x_{1002}^2 + y_{1002}^2 \neq 0$, then the equilibrium position is unstable.

3°. For $\omega_1 = 3\omega_2$ the Hamiltonian can be reduced to the form

$$H = 3\omega_2 r_1 - \omega_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + \\ + {}^1/_3\omega_2\sqrt{3(x_{1003}^2 + y_{1003}^2)}\, r_2\sqrt{r_1r_2}\sin(\varphi_1 + 3\varphi_2) + O((r_1 + r_2)^{5/2}) \tag{2.8}$$

The constants $c_{20},\ c_{11},\ c_{02},\ x_{1003},\ y_{1003}$ in (2.8) depend on the coefficients of the forms $H_2,\ H_3,\ H_4$. The equilibrium position is instable if the inequalities

$$x_{1003}^2 + y_{1003}^2 \neq 0, \qquad 3\omega_2\sqrt{x_{1003}^2 + y_{1003}^2} > |c_{20} + 3c_{11} + 9c_{02}|$$

are fulfilled.

4°. Now let us consider the stability of the equilibrium position when condition (2.4) is not fulfilled.

If $k_1\omega_1 + k_2\omega_2 \neq 0$ for integers k_1 and k_2 which satisfy the condition $0 < |k_1| + |k_2| \leqslant 2m$, then an analytical canonical substitution of variables allows us to reduce [10] Hamiltonian (2.2) to the form

$$H = \omega_1 r_1 - \omega_2 r_2 + \sum_{i+j=2}^{m} c_{ij} r_1^i r_2^j + H^*(r_1, r_2, \varphi_1, \varphi_2)$$

$$(H^* = O((r_1 + r_2)^{m+1/2})) \tag{2.9}$$

where r_i, φ_i can be determined from formulas (2.7), and the function H^* has the period 2π in φ_1 and φ_2.

The coefficients c_{ij} are the invariants of Hamiltonian (2.2) relative to canonical transformations. Let us consider the polynomial

$$h(\varepsilon) \equiv \sum_{i+j=2}^{m} c_{ij} \omega_1^j \omega_2^i \varepsilon^{i+j} \tag{2.10}$$

If $h(\varepsilon) \not\equiv 0$, then we say that the general elliptic case holds. In the Arnol'd theorem the inequality $h(\varepsilon) \not\equiv 0$ arises from the coefficient of ε^2 in polynomial (2.10). If this coefficient is equal to zero, i.e. if condition (2.4) is not fulfilled, then coefficients of higher powers of ε in polynomial (2.10) must be obtained. Requirements concerning the absence of resonance more rigid than (2.3) must then be imposed on ω_1 and ω_2.

The first nonzero coefficient of polynomial (2.10) is that of ε^m. The following theorem is then valid.

Theorem 2.1. Let Hamiltonian (2.2) satisfy the following conditions:

1) the characteristic equation of the system with the Hamiltonian H_2 has the purely imaginary roots $\pm i\omega_1$, $\pm i\omega_2$;

2) the frequencies ω_1 and ω_2 satisfy the inequalities

$$k_1\omega_1 + k_2\omega_2 \neq 0 \quad \text{for } 0 < |k_1| + |k_2| \leqslant 2m \tag{2.11}$$

3) the inequality

$$\sum_{i=0}^{m} c_{m-i,\,i} \omega_1^i \omega_2^{m-i} \neq 0 \tag{2.12}$$

is fulfilled.

The equilibrium position is then stable.

Let us outline the proof of this Theorem. The first step is to reduce Hamiltonian (2.2) to the form (2.9), and, using the integral $H = \text{const}$, to reduce system (2.1) to a system with one degree of freedom [4]. Applying Moser's theorem on invariant curves [12] to the mapping generated by the resulting Hamiltonian system of differential equations [13], we can show that fulfillment of the above theorem at each level $H = \text{const}$ in any neighborhood of the origin ensures the existence of two-dimensional invariant tori of system (2.1). This implies the stability of the equilibrium position. Similar applications of Moser's theorem to dynamics problems can be found, for example, in [14 15].

3. Let us prove the following theorem on the stability of the libration points.

Theorem 3.1. The triangular libration points are stable for all ratios of the masses in the range $(m_1 + m_2)^2 > 27 m_1 m_2$ except for the ratios

$$\frac{m_1}{m_2} = \frac{643 + 15\sqrt{1833}}{32}, \qquad \frac{m_1}{m_2} = \frac{73 + 5\sqrt{213}}{2}$$

for which they are unstable.

Let us introduce a coordinate system which rotates with the angular velocity n, whose origin coincides with the center of mass O of the bodies S and J, and whose x-axis coincides with the straight line OJ. Denoting the coordinates of the point P in this system by x, y and the projections on the axes x, y of the velocity of P relative to the fixed coordinate system by u, v we can write the Hamiltonian of the problem in the form [4]

$$H = \frac{1}{2}(u^2 + v^2) + n(uy - vx) - \frac{m_1}{SP} - \frac{m_2}{JP} \tag{3.1}$$

Let us denote the length of SJ by l. Then, as we know, $n^2 l^3 = m_1 + m_2$, and the solution corresponding to the triangular libration point for the system of equations with Hamiltonian (3.1) is the equilibrium position [4]

$$x = a, \quad y = b, \quad u = -nb, \quad v = na \tag{3.2}$$

where

$$a = \frac{m_1 - m_2}{m_1 + m_2}\frac{l}{2}, \qquad b = \frac{\sqrt{3}}{2}l$$

On substituting variables according to the expressions

$$x = a + q_1 l, \quad y = b + q_2 l, \quad u = -nb + p_1 nl, \quad v = na + p_2 nl, \quad \tau = nt$$

we can write the solution under investigation as $q_1 = q_2 = p_1 = p_2 = 0$.

Expanding Hamiltonian (3.1) in the neighborhood of the origin $q_1 = q_2 = p_1 = p_2 = 0$ in a series in powers of q_i, p_i, we obtain

$$H = H_2 + H_3 + H_4 + \ldots + H_k + \ldots \tag{3.3}$$

$$\begin{aligned}
H_2 &= \tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_2^2 + q_2 p_1 - q_1 p_2 + \tfrac{1}{8} q_1^2 - k q_1 q_2 - \tfrac{5}{8} q_2^2 \\
H_3 &= -\frac{7\sqrt{3}k}{36} q_1^3 + \frac{3\sqrt{3}}{16} q_1^2 q_2 + \frac{11\sqrt{3}k}{12} q_1 q_2^2 + \frac{3\sqrt{3}}{16} q_2^3 \\
H_4 &= \frac{37}{128} q_1^4 + \frac{25k}{24} q_1^3 q_2 - \frac{123}{64} q_1^2 q_2^2 - \frac{15k}{8} q_1 q_2^3 - \frac{3}{128} q_2^4 \\
H_5 &= \frac{23\sqrt{3}k}{576} q_1^5 - \frac{285\sqrt{3}}{256} q_1^4 q_2 - \frac{215\sqrt{3}k}{288} q_1^3 q_2^2 + \frac{345\sqrt{3}}{128} q_1^2 q_2^3 + \\
&\quad + \frac{555\sqrt{3}k}{576} q_1 q_2^4 - \frac{33\sqrt{3}}{256} q_2^5 \\
H_6 &= -\frac{331}{1024} q_1^6 + \frac{49k}{128} q_1^5 q_2 + \frac{6105}{1024} q_1^4 q_2^2 - \frac{35k}{64} q_1^3 q_2^3 - \frac{7965}{1024} q_1^2 q_2^4 - \\
&\quad - \frac{119k}{128} q_1 q_2^5 + \frac{383}{1024} q_2^6 \\
k &= 3\sqrt{3}(m_1 - m_2) / 4(m_1 + m_2)
\end{aligned} \tag{3.4}$$

We have omitted the constant term in (3.3). The condition of stability in the first approximation can be written in the form of inequalities,

$$^{23}/_{16} < k^2 < ^{27}/_{16} \tag{3.5}$$

The frequencies of the oscillating system with the Hamiltonian satisfy the equation

$$\omega^4 - \omega^2 + (^{27}/_{16} - k^2) = 0 \tag{3.6}$$

We can assume that $\omega_1 > \omega_2 > 0$. in range (3.5).

By making use of the Arnol'd theorem we can verify that the equilibrium position $q_1 = q_2 = p_1 = p_2 = 0$ is stable in the range (3.5) except possibly in those cases where one of the equations

$$\omega_1 = 2\omega_2, \quad \omega_1 = 3\omega_2, \quad c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 = 0$$

is fulfilled.

Computations show that

$$\omega_1 = 2\omega_2 = \frac{2\sqrt{5}}{5} \quad \text{for} \quad k^2 = \frac{611}{400}\left(\frac{m_1}{m_2} = \frac{643 + 15\sqrt{1833}}{32}\right)$$

$$\omega_1 = 3\omega_2 = \frac{3\sqrt{10}}{10} \quad \text{for} \quad k^2 = \frac{639}{400}\left(\frac{m_1}{m_2} = \frac{73 + 5\sqrt{213}}{2}\right)$$

$$c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 = 0 \quad \text{for} \quad k^2 = 1.6146\ (m_1/m_2 = 90.6282)$$

The Arnol'd theorem is inapplicable for the three indicated mass ratios. Let us make use of the results of Sect. 2 to complete our solution of the stability problem.

For $\omega_1 = 2\omega_2$ Hamiltonian (3.3) is reducible to the form (2.6), where

$$x_{1002}^2 + y_{1002}^2 = 4.108 \neq 0$$

The equilibrium position is unstable.

For $\omega_1 = 3\omega_2$ the Hamiltonian is reducible to the form (2.8). In this case

$$3\omega_2\sqrt{x_{1003}^2 + y_{1003}^2} = 23.2826, \qquad |c_{20} + 3c_{11} + 9c_{02}| = 4.1705$$

and the equilibrium position is also unstable.

Now let us consider the ratio $m_1/m_2 = 90.6282$ when condition (2.4) of the Arnol'd theorem is not fulfilled. Computations show that for Hamiltonian (3.3) reduced to the form (2.9) we have

$$\omega_1 = 0.9596, \quad \omega_2 = 0.2813, \quad c_{20} = 0.0978, \quad c_{11} = -1.3892, \quad c_{02} = 0.3988$$
$$c_{30} = -0.2193, \quad c_{21} = 7.7942, \quad c_{12} = -209.9311, \quad c_{03} = -14.5289$$

Verifying inequalities (2.11) and (2.12) for $m = 3$, we find that

$$\omega_1 \neq \omega_2, \quad \omega_1 \neq 2\omega_2, \quad \omega_1 \neq 3\omega_2, \quad \omega_1 \neq 4\omega_2, \quad \omega_1 \neq 5\omega_2, \quad 2\omega_1 \neq 3\omega_2$$
$$c_{30}\omega_2^3 + c_{21}\omega_2^2\omega_1 + c_{12}\omega_2\omega_1^2 + c_{03}\omega_1^3 = -66.6312 \neq 0$$

All of the conditions of Theorem 2.1 are fulfilled, and the equilibrium position is stable.

The author is grateful to V.A. Sarychev for his interest in the present study.

BIBLIOGRAPHY

1. Lagrange, J.L., Essais sur le probleme des trois corps. Paris, 1772.
2. Gascheau, G., Comptes Rendus, Vol.16, p.393, 1843.
3. Duboshin, G.N., Celestial mechanics. Analytical and Qualitative Methods. Moscow, "Nauka", 1964.
4. Whittaker, E.T., Analytical mechanics. Moscow-Leningrad, ONTI, 1937.
5. Arnol'd, V.I., On the stability of the equilibrium positions of a Hamiltonian system of ordinary differential equations in the general elliptic case. Dokl. Akad. Nauk SSSR, Vol.137, No.2, 1961.
6. Leontovich, A.M., On the stability of the Lagrangian periodic solutions of the bounded three-body problem. Dokl. Akad. Nauk SSSR, Vol.143, No.3, 1962.
7. Deprit-Bartholome, A., Stability of the triangular Lagrangian points. Astronom. J., Vol.72, No.2, 1967.
8. Arnol'd, V.I., Small denominators and problems of motion stability in classical and celestial mechanics. Usp. Mat. Nauk, Vol.18, No.6, 1963.
9. Malkin, I.G., The Theory of Stability of Motion. Moscow, "Nauka", 1941.
10. Birkhoff, D.D., Dynamic Systems. Moscow, Gostekhteorizdat, 1941.
11. Markeev, A.P., Stability of a canonical system with two degrees of freedom in the presence of resonance. PMM Vol.32, No.4, 1968.
12. Moser, U., Curves invariant under area-preserving ring mappings. Moscow, Izd. inostr. lit., Collected papers: "Mathematics", Vol.6, No.5, 1962.
13. Siegel, K.L., Lectures on Celestial Mechanics. Moscow, Izd. inostr. lit., 1959.
14. Kyner, W.T., Qualitative properties of orbits about an oblate planet. Comm. Pure Appl. Mat., Vol.17, No.2, 1964.
15. Proctor, T. and Strabl, R., The motion of two weakly interacting nonlinear oscillators. Moscow, Izd. inostr. lit., Collected Papers: "Mechanics", No.2, 1966.

Translated by A.Y.

THE PROBLEM OF SMALL MOTIONS OF A BODY WITH A CAVITY PARTIALLY FILLED WITH A VISCOUS FLUID

PMM Vol. 33, No. 1, 1969, pp. 117-123

S.G. KREIN, NGO ZUI KAN
(Voronezh)

(Received May 16, 1968)

The general problem of motion about a fixed point O_1 of a rigid body with a cavity partially or totally filled with a viscous incompressible fluid, under the action of gravity is studied here in its linearized approximation. Surface tension is neglected.

For the case of motion about the center of mass when the cavity is completely filled, this problem was considered in [1]. The general problem when the fluid viscosity is assumed to be small was considered in the paper of F.L. Chernous'ko [2].

1. Equations of motion of the fluid. We denote by Ω the region (in a moving coordinate system O_1xyz rigidly attached to the body) which is filled with the undisturbed fluid. We denote by Γ_0 the undisturbed free surface of the fluid, and by Γ_1 that part of the wall of the cavity in contact with the fluid. In the linearized approximation to the Navier-Stokes equations, the fluid motion is described in the O_1xyz system by